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TIME EVOLUTION AND FLUCTUATIONS OF THE PROBABILITY DENSITY AND ENTROPY FUNCTION FOR A CLASS OF NONLINEAR STOCHASTIC SYSTEMS IN MATHEMATICAL PHYSICS

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Abstract—A mathematical theory is proposed to describe the time evolution and fluctuations for the probability density and the entropy functions related to the state variable of nonlinear stochastic systems. The class of physical systems whose evolution can be modelled by nonlinear stochastic ordinary differential equations is considered and the evolution equation for the above specified functions is derived under suitable regularity conditions. Quantitative results in the applications are supplied on the basis of an analysis using the Adomian's decomposition method.

1. INTRODUCTION

Mathematical models describing the evolution of real physical systems in mathematical physics can be formulated in terms of nonlinear ordinary vector differential equations. For a large class of physical systems, it is reasonable to assume, after the identification process, that initial conditions and parameters involve some uncertainty or random fluctuations. The deterministic case is regarded only as a limit of the stochastic one. Moreover, one should keep in mind that any parameter characterizing the model has to be based upon observation of the real system without any *a priori* restriction. In other words, real systems should not be modelled by a preconceived or an *a priori* stochastic process.

The contribution of Bellman to the development of stochastic analysis and mathematical methods for the analysis of stochastic systems in mathematical physics is well recognized. Let us recall the paper by Bellman and Kalaba[1] on wave propagation in a random media followed, for example, by those quoted as Refs. 2 and 3. More recently, Bellman showed considerable interest in Adomian's "decomposition" method, which we use in this paper, and encouraged a series of books[4–6]. The Adomian method is documented in a recently published book[4] and developed, starting from Ref. 7, through various contributions for which an extensive bibliography appears in Ref. 4. Moreover, various papers can be found showing the cooperation between Bellman and Adomian such as the paper on the stochastic Riccati equation[8], on the Ito equation[9], on biological systems[10] and on the "decomposition" method as a mathematical method to deal with partial differential equations[11].

Some of the previously quoted results will be utilized in this paper, which deals with the class of physical systems whose evolution in time can be described by stochastic nonlinear ordinary differential equations and which proposes an analysis of the time evolution, including the related fluctuations, both of the probability density and of the entropy function of the physical state of the class of systems considered. The starting point of the analysis consists of results, obtained and developed in some previous papers[12–17], on the probability density as a function of the state variable of differential equations with random initial conditions and parameters.

The mathematical framework is defined in the second section. The third section introduces the concept of fluctuations of the probability density and derives, under suitable regularity conditions, the evolution equation governing the fluctuations. An analogous analysis on the evolution and fluctuations of the entropy function is also realized in the same section. The fourth section proposes an analysis of the Adomian method in order to supply quantitative results for the above class of stochastic systems. More precisely on this point, the decomposition method can be used in order to obtain analytical solutions for all the evolution equations proposed in Sec. 3, so that quantitative results, as documented in the same section, can be obtained on the time evolution both for the probability density and entropy functions as well as for the time evolution of their fluctuations.

2. DESCRIPTION OF THE MATHEMATICAL MODEL

Let us consider a complete probability space $(\Omega, \mathfrak{B}, p)$, where Ω is the abstract space of the elementary events ω; $\mathfrak{B}$ is a σ-field of the subsets of Ω, and p is the probability density which induces a probability measure on $\mathfrak{B}$ such that

$$\forall E : \mu(E) = \int_E p(\omega)\, d\omega. \tag{1}$$

We consider the class of physical systems such that their time evolution can be defined by a mathematical model described by the following set of axioms:

(A.1) The state of the system is defined by the variable $\mathbf{x} : \Omega \cdot I \to D \subset R^n$, where $I = [0, t] \subseteq R$ is the domain of the independent variable t and D is an open bounded subset of R^n.

(A.2) The time evolution of the system is defined by an ordinary stochastic differential equation of the class

$$x_i(\omega, t) = x_{0i}(\omega) + \int_I f_i(\mathbf{x}(\omega, s), s, \mathbf{r}(\omega, s))\, ds, \quad i = 1, \ldots, n, \tag{2}$$

where $\mathbf{x} = \{x_i\}^T$, with $x_{0i}(\omega)$ a random initial condition of the component x_i of $\mathbf{x}$:

$$\mathbf{x}_0(\omega) = \mathbf{x}(\omega, t = 0) = \{x_{0i}(\omega)\}^T : \Omega \longrightarrow D_0 \subset R^n.$$

In Eq. (2) the f_i are deterministic functions of $\mathbf{x}$, $\mathbf{r}$ and t, differentiable with respect to all their arguments, and $\mathbf{r} = \{r_h\}^T$ a set of known, bounded stochastic processes r_h defined for $t \in I$ over the complete probability space $(\Omega, \mathfrak{B}, p)$ and belonging to one of the following classes:

$$r_h = \sum_k \alpha_{hk}(\omega)\varphi_{hk}(t), \quad \boldsymbol{\alpha}(\omega) = \{\alpha_{hk}\} : \Omega \longrightarrow A \subset R^q, \tag{3a}$$

where $\alpha_{hk}(\omega)$ are known random variables and φ_{hk} known deterministic functions of t, or

$$r_h = \sum_n \alpha_{nh}(\omega) I_n\{N(t) = n\}, \tag{3b}$$

where $\alpha_{nh}(\omega)$ are piecewise constant random variables in the deterministic sequence of time intervals I_n, or finally,

$$r_h = r_h(\mathbf{x}; \boldsymbol{\alpha}(\omega)), \tag{3c}$$

where r_h are known differentiable functions of $\mathbf{x}$ and of a set $\boldsymbol{\alpha}$ of known real-valued random variables.

(A.3) There exist sufficient conditions such that for $t \in I$ and $\forall \mathbf{x}_0$, $\boldsymbol{\alpha}$, a unique solution $\mathbf{x}(t; \mathbf{x}_0, \boldsymbol{\alpha})$ exists for Eq. (2).

It is worth pointing out that the axiomatization is very general; however, in the mathematical modelling of real physical systems, only a limited number of random parameters characterize the model in general.

After these preliminaries, and recalling that the concepts proposed in this section will be clarified in the application considered in the fourth section, the concept of statistics on the probability density is now introduced. Consider then the map $\mathbf{x}_0(\omega) \to \mathbf{x}_t(\omega, t)$. If $P_0(\mathbf{x}_0)$ is the probability density joined to the variable $\mathbf{x}_0(\omega)$, then the map $\mathbf{x}_0 \to \mathbf{x}_t$ can define, under suitable regularity conditions[13], the following map on the probability density:

$$P_0(\mathbf{x}_0) \longrightarrow P_t(\mathbf{x}; t, \boldsymbol{\alpha}), \tag{4}$$

where in P_t time t and $\boldsymbol{\alpha}$ are regarded as parameters. Therefore, some statistics on P_t can be realized. For instance its mean value $\forall \mathbf{x}$, t is given by

$$\langle P_t \rangle = \int_A P_t(\mathbf{x};\, t,\, \boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}, \tag{5}$$

whereas the fluctuations of the probability density from its mean value can be quantitatively indicated by its variance:

$$V(P_t) = \int_A [P_t(\mathbf{x};\, t,\, \boldsymbol{\alpha}) - \langle P_t(\mathbf{x};\, t)\rangle]^2 P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}. \tag{6}$$

This kind of statistics, which can be extended naturally to the components of $\mathbf{x}$, can be crucial for those physical systems where inner fluctuations in the system itself give rise to fluctuations in the time evolution of the state variable. As already mentioned, the examples of Sec. 4 have also the objective of demonstrating a clarification of the above concepts.

The objective of this research is the analysis of the mathematical problems (existence, continuity and constructive methods) related to the statistics of the map: $P_0 \rightarrow P_t$ and the analysis of the applications to the mathematical models considered in the fourth section.

3. TIME EVOLUTION AND STATISTICS ON THE PROBABILITY DENSITY

The assumption (A.3) of Sec. 2 assures, for every fixed value of $\boldsymbol{\alpha} \in A$, the uniqueness of the map $\mathbf{x}_0 \rightarrow \mathbf{x}_t$. This map will be indicated by $\Phi_{\alpha t}$:

$$\mathbf{x}_t = \Phi_{\alpha t}\mathbf{x}_0 : \Omega \cdot I \longrightarrow D \subset \mathrm{R}^n \tag{7}$$

and defines a diffeomorphism. The map $\Phi_{\alpha t}$ also defines the image probability space $[\mathbf{x}_t(\Omega), \mathbf{x}_t(\mathfrak{B}), P_t]$ of $(\Omega, \mathfrak{B}, p)$. Recall (see Ref. 12 and the propositions 1–3 of Sec. 3 in Ref. 15) that if the time evolution of the probability measure μ_t is absolutely continuous with respect to the initial measure

$$\mu_t(\Phi_{\alpha t}C) > 0 \Rightarrow \mu_0(\Phi_{\alpha t}^{-1}C) > 0, \tag{8}$$

where C is a Borel subset of R^n belonging to the σ-algebra $\mathbf{x}_t(\mathfrak{B})$, then the time evolution of the probability density P_t is governed by the Jacobian $J(\mathbf{x}_t \rightarrow \mathbf{x}_0) = J(\mathbf{x}_t \rightarrow \Phi_{\alpha t}^{-1}\mathbf{x}_t)$ of the inverse mapping

$$P_t(\mathbf{x}_t = \Phi_{\alpha t}\mathbf{x}_0;\, t) = J(\mathbf{x}_t \longrightarrow \Phi_{\alpha t}^{-1}\mathbf{x}_t) P_0(\mathbf{x}_0), \tag{9}$$

where J satisfies the differential equation

$$J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) = 1 - \int_0^t J(s;\, \mathbf{x}_0,\, \alpha) g(\mathbf{x}(s;\, \mathbf{x}_0,\, \alpha);\, s,\, \alpha)\, \mathrm{d}s, \tag{10}$$

where

$$g = \nabla \cdot \mathbf{f} = \sum_i \partial f_i / \partial x_i. \tag{11}$$

The result allow both the calculation of the probability density $P_t = P_t(\mathbf{x};\, t,\, \boldsymbol{\alpha})$ and the statistics previously indicated by Eqs. (5, 6). In fact, Eqs. (9, 11) allow the construction of P_t, whereas its mean value is given by

$$\langle P_t \rangle \equiv E\{P_t(\mathbf{x}_t = \Phi_{\alpha t}\mathbf{x}_0;\, t,\, \alpha)\} = P_0(\mathbf{x}_0) \int_A J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}, \tag{12}$$

and the variance

$$V(P_t) = \int_A [P_0(\mathbf{x}_0)J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) - \langle P_t\rangle]^2 P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} = P_0^2(\mathbf{x}_0)\langle J^2\rangle - \langle P_t\rangle^2, \tag{13}$$

where

$$\langle J^2\rangle = \int_A J^2(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}. \tag{13a}$$

Following this line, the above results can now supply an expression and an evolution equation for some entropy definitions related to the class of stochastic systems considered. In particular, the Boltzmann–Shannon–Gross (BSG) generalized entropy function is considered here. According to Ref. 18, the following definition is given:

DEFINITION

If, at given $\boldsymbol{\alpha}$ and for the class of systems defined in Eq. (2), the probability measure μ_t referred to the state variable $\mathbf{x}$ is absolutely continuous with respect to some initial probability measure μ_0, i.e. $\mu_t \ll \mu_0$, then the Radon–Nikodym derivative $\mathrm{d}\mu_t/\mathrm{d}\mu_0$ exists and the following expression of the BSG entropy function can be defined:

$$S(t;\, \boldsymbol{\alpha}) \propto -\int (\mathrm{d}\mu_t/\mathrm{d}\mu_0)\ln(\mathrm{d}\mu_t/\mathrm{d}\mu_0)\, \mathrm{d}\mu_0. \tag{14}$$

The time evolution of the mean value and fluctuations of the above defined entropy function can now be studied (see Refs. 12 and 15). If for every $\boldsymbol{\alpha} \in A$ there exist sufficient conditions to assure the absolute continuity of μ_t with respect to μ_0, then the Radon–Nikodym derivative is given by the Jacobian of the inverse mapping

$$\mathrm{d}\mu_t/\mathrm{d}\mu_0 = J(\mathbf{x}_t \longrightarrow \Phi_{\alpha t}^{-1}\mathbf{x}_t) \tag{15}$$

(see Ref. 15, Proposition 3). Then, casting Eq. (15) into Eq. (14) and averaging over $\boldsymbol{\alpha}$, the following "mean" BGS entropy function is obtained:

$$\langle S(t)\rangle = \int_A S(t;\, \boldsymbol{\alpha})P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \propto -\int_{A\cdot D_0} J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})\ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}\, \mathrm{d}\mu_0. \tag{16}$$

Considering now that $\mathrm{d}\mu_0 = P_0(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0$ and that $\dot{J} = -gJ$ [see Eqs. (10 and 11)], time differentiation of Eq. (16) yields the following evolution equation for the "mean" entropy function:

$$\dot{\langle S(t)\rangle} = \mathrm{d}\langle S(t)\rangle/\mathrm{d}t \propto -\int_{A\cdot D_0} J(t,\, \mathbf{x}_0,\, \boldsymbol{\alpha})g(\mathbf{x}(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha});\, t,\, \boldsymbol{\alpha}) \times [1 + \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})]P(\boldsymbol{\alpha})P_0(\mathbf{x}_0)\, \mathrm{d}\boldsymbol{\alpha}\, \mathrm{d}\mathbf{x}_0. \tag{17}$$

In the same fashion as for the "mean" entropy, one can evaluate the time evolution of the BSG entropy fluctuations which are given by its variance, defined as

$$V(S) = \int_A [S(t;\, \boldsymbol{\alpha}) - \langle S(t)\rangle]^2 P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} = \langle S^2(t)\rangle - \langle S(t)\rangle^2, \tag{18}$$

where

$$\langle S^2(t)\rangle = \int_A S^2(t;\, \boldsymbol{\alpha})P(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}. \tag{19}$$

Casting Eqs. (14–16) into Eq. (18), the variance can be calculated as

$$V(S) \propto \int_A \left[\int_{D_0} J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) P_0(\mathbf{x}_0)\, d\mathbf{x}_0 \right]^2 P(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} - \left[\int_{A \cdot D_0} J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) P(\boldsymbol{\alpha}) P_0(\mathbf{x}_0)\, d\boldsymbol{\alpha}\, d\mathbf{x}_0 \right]^2, \quad (20)$$

whereas time differentiation of the above equation supplies the following evolution equation for the variance of the BSG entropy:

$$\dot{V} = dV(S)/dt \propto 2 \int_A [S(t;\, \boldsymbol{\alpha}) - \langle S(t) \rangle][\dot{S}(t;\, \boldsymbol{\alpha}) - \dot{\langle S(t) \rangle}] P(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}, \quad (21)$$

and performing standard calculations

$$\dot{V} \propto 2[\langle S(t)\dot{S}(t) \rangle - \langle S(t) \rangle \dot{\langle S(t) \rangle}] = 2 \operatorname{cov}(S,\, \dot{S}), \quad (22)$$

where $\langle S(t) \rangle$ and $\dot{\langle S(t) \rangle}$ are supplied by Eqs. (16) and (17), respectively, and

$$\langle S(t)\dot{S}(t) \rangle = \int_A S(t;\, \boldsymbol{\alpha}) \dot{S}(t;\, \boldsymbol{\alpha}) P(\boldsymbol{\alpha})\, d\boldsymbol{\alpha} \propto - \int_{A \cdot D_0} J^2(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) g(\mathbf{x}(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha});\, t,\, \boldsymbol{\alpha}) \ln J(t;\, \mathbf{x}_0,\, \dot{\boldsymbol{\alpha}}) \times [1 + \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})] \cdot P(\boldsymbol{\alpha}) P_0(\mathbf{x}_0)\, d\boldsymbol{\alpha}\, d\mathbf{x}_0. \quad (23)$$

Monotonicity conditions for the "mean" entropy function can be stated by considering the properties of the function $g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})$ defined by Eq. (11), as proved by the following.

THEOREM

(i) If $g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) < 0$ for every $\mathbf{x}_0 \in D_0$ and $\boldsymbol{\alpha} \in A$, then the "mean" BSG entropy function is monotonically decreasing in $I = [0,\, t]$.

(ii) If, for every $\mathbf{x}_0 \in D_0$ and $\boldsymbol{\alpha} \in A$, the following conditions hold:

$$g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) > 0, \quad (24)$$

$$\int_0^t g(s;\, \mathbf{x}_0,\, \boldsymbol{\alpha})\, ds < 1, \quad (24a)$$

then the "mean" entropy function is monotonically increasing in I.

Proof. Since the solution of the evolution Eq. (10) for the Jacobian is

$$J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) = \exp\left[- \int_0^t g(s;\, \mathbf{x}_0,\, \boldsymbol{\alpha})\, ds \right], \quad t \in I, \quad (25)$$

then the condition $g < 0$ implies $J > 1$ and, moreover,

$$J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})[1 + \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})] < 0 \quad (26)$$

for each realization of $\mathbf{x}_0 \in D_0$ and $\boldsymbol{\alpha} \in A$. Considering now Eq. (15) which defines the evolution of the "mean" entropy, and remembering the positivity properties of the probability densities $P_0(\mathbf{x}_0)$ and $P(\boldsymbol{\alpha})$, it follows for every $t \in I$ that

$$\int_{A \cdot D_0} J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})[1 + \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})] P(\boldsymbol{\alpha}) P_0(\mathbf{x}_0)\, d\boldsymbol{\alpha}\, d\mathbf{x}_0 < 0,$$

which proves the first part of the theorem. On the other hand, if both the conditions (24) and (24a) hold, it follows from Eq. (25) that

$$0 < J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) < 1, \quad 0 > -\int_0^t g(s;\, \mathbf{x}_0,\, \boldsymbol{\alpha})\, ds = \ln J(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha}) > -1,$$

and therefore the left-side member of Eq. (26) is now positive for each realization of $\mathbf{x}_0$ and $\boldsymbol{\alpha}$. Then, multiplying this expression by $P_0(\mathbf{x}_0)P(\boldsymbol{\alpha})$ and integrating over $\mathbf{x}_0$ and $\boldsymbol{\alpha}$, one obtains from Eq. (15) that $\langle \dot{S} \rangle > 0$ for every $t \in I$, which proves the part (ii) of the theorem.

With regard to the second-order moments of entropy, it must be observed that the conditions on the function $g(t;\, \mathbf{x}_0,\, \boldsymbol{\alpha})$ which have been introduced in the above theorem (or analogous ones) are not sufficient to define monotonicity properties for the variance of the BSG entropy function. As it is stated by Eq. (22), the monotonicity of $V(S)$ is related to the sign of the covariance between the entropy $S(t;\, \boldsymbol{\alpha})$ and its first derivative with respect to the time.

4. METHOD AND APPLICATIONS IN MECHANICS

The recovering of the relevant quantities defined in the preceding section, i.e. probability density, entropy and their related fluctuations, can be realized after a quantitative analysis of the evolution equations defined in Sec. 3 and, in particular, after solution of Eq. (2). However, considering that this equation will be, in general, a nonlinear one, the problem shows all the inner difficulties of nonlinear stochastic analysis. Besides the numerical treatment by the stochastic Runge–Kutta integration procedure as documented in Ref. 19, Adomian's decomposition method is valuable in order to obtain analytical expressions for the solutions of Eq. (2) and consequently of the various evolution equations defined in Sec. 3.

Keeping this in mind, consider Eq. (2) which can also be rewritten in the following operator form:

$$x_i = x_{0i} + \lambda \mathcal{L}^{-1} f_i, \quad \lambda = 1, \quad \mathcal{L}^{-1} = \int_0^t (\cdot)\, ds, \tag{27}$$

and denote by $L^2(\Omega) = L^2(\Omega,\, \mathfrak{B},\, p)$ the space of the second-order real valued functions of random variables on Ω. Then the scalar product of the probabilized Hilbert space is

$$\langle f(\mathbf{x}(\omega)),\, g(\mathbf{x}(\omega))\rangle_\Omega = \int f(\mathbf{x})g(\mathbf{x})P(\mathbf{x})\, d\mathbf{x} = \langle f,\, g\rangle_\Omega, \tag{28}$$

and the induced norm

$$\|f\|_{2\Omega} = \left[\int f^2(\mathbf{x})P(\mathbf{x})\, d\mathbf{x} \right]^{1/2} = \langle f,\, f\rangle_\Omega^{1/2}. \tag{29}$$

In this framework the following additional assumptions are stated.

(H.1) The functions f_i are m-differentiable with respect to all the components of $\mathbf{x}$, and all derivatives are bounded by some constants

$$\sup_{\substack{i=1,\ldots,n \\ h=1,\ldots,m}} |\partial^h f_i / \partial x_1 \ldots \partial x_h| < M. \tag{30}$$

The assumption above will be afterwards utilized in order to supply some convergence criteria. Consider now the following decomposition:

$$x_i \simeq \sum_{h=0}^{m} \lambda^h x_i^{(h)}, \tag{31}$$

$$f_i = \sum_{h=0}^{m} \lambda^h f_i^{(h)}, \quad f_i^{(0)} = f_i(\mathbf{x}^{(0)}; t, \boldsymbol{\alpha}),$$

$$f_i^{(h)} = \frac{1}{h!} \cdot \frac{\mathrm{d}^h f_i}{\mathrm{d}\lambda^h}\bigg|_{\lambda=0} = f_i^{(h)}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(h)}; t, \boldsymbol{\alpha}), \quad h = 1, 2, \ldots, m. \tag{32}$$

According to (H.1), the expansion (32) can be formally realized. Moreover, the following additional hypothesis is assumed.

(H.2) There exist sufficient conditions such that in some bounded time interval, $t \in I$, the following holds:

$$m \longrightarrow \infty \Rightarrow \left\| f_i - \sum_h \lambda^h f_i^{(h)} \right\|_{2\Omega} \longrightarrow 0,$$

for $\lambda \in [0, 1]$ and for each realization of $\mathbf{x}_0$ and $\boldsymbol{\alpha}$.

Hypothesis (H.2) is related to (H.1). In fact, Eq. (32) can be regarded as a MacLaurin power expansion of λ about $\lambda = 0$, and after (H.1) the remainder is bounded by $\lambda^m M/m!$, then (H.2) holds.

Consider now Eqs. (31 and 32) and cast them into Eq. (27). Equating the terms with the same power of λ yields the following sequence of equations:

$$\mathbf{x}^{(0)} = \mathbf{x}_0, \tag{33a}$$

$$\mathbf{x}^{(1)} = \mathcal{L}^{-1}\mathbf{f}(\mathbf{x}_0; t, \boldsymbol{\alpha}), \tag{33b}$$

$$\mathbf{x}^{(2)} = \mathcal{L}^{-1}\mathbf{f}^{(1)}(\mathbf{x}_0, \mathbf{x}^{(1)}; t, \boldsymbol{\alpha}), \tag{33c}$$

$$\mathbf{x}^{(m)} = \mathcal{L}^{-1}\mathbf{f}^{(m-1)}(\mathbf{x}_0, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m-1)}; t, \boldsymbol{\alpha}). \tag{33d}$$

Then each equation is solved with a quadrature taking into account the result of the preceding one, the final result being

$$\mathbf{x} = \mathbf{x}_0 + \sum_{h=1}^{m} \mathcal{L}^{-1}\mathbf{f}^{(h-1)}(\mathbf{x}_0, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(h-1)}; t, \boldsymbol{\alpha}). \tag{34}$$

Often the quadrature can be executed analytically. Otherwise, an augmented variable can be set:

$$\mathbf{y} = \{\mathbf{x}_0, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}\}^T \in R^{n \cdot m}, \tag{35}$$

which satisfies the vector equation

$$\mathbf{y} = \mathbf{y}_0 + \mathcal{L}^{-1}\mathbf{F}(\mathbf{y}; t, \boldsymbol{\alpha}), \tag{36}$$

where $\mathbf{y}_0 = \mathbf{x}_0$ and $\mathbf{F} = \{\mathbf{f}^{(0)}, \ldots, \mathbf{f}^{(m-1)}\}^T$. Then the various quadratures of the sequence can be executed simultaneously with standard numerical methods. The hypothesis (H.2) assures convergence for large values of m.

The calculation of the probability density and of the entropy function is a consequence. In fact, Eq. (10) and its formal solution defined in Eq. (25) can be considered together with the solution (34). Then the actual expression of J is simply given by a quadrature:

$$J(t; \mathbf{x}_0, \boldsymbol{\alpha}) = \exp\left\{ - \int_0^t g\left(\mathbf{x}_0 + \sum_{h=1}^{m} \mathcal{L}^{-1}\mathbf{f}^{(h-1)}(\mathbf{x}_0, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(h-1)}; s, \boldsymbol{\alpha})\right) \mathrm{d}s \right\}. \tag{37}$$

Alternatively, an approximated expression of J can be recovered by considering the decompositions

$$J \simeq \sum_{h=0}^{m} \lambda^h J^{(h)}, \tag{38}$$

$$\begin{aligned} g(\mathbf{x}(t; \mathbf{x}_0, \boldsymbol{\alpha}); t, \boldsymbol{\alpha}) &\simeq \sum_{h=0}^{m} \lambda^h g^{(h)}, \quad g^{(0)} = g(\mathbf{x}_0; t, \boldsymbol{\alpha}), \\ g^{(h)} &= \frac{1}{h!} \cdot \frac{\mathrm{d}^h g}{\mathrm{d}\lambda^h}\bigg|_{\lambda=0} = g^{(h)}(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(h)}; t; \boldsymbol{\alpha}), \\ h &= 1, 2, \ldots, m, \end{aligned} \tag{39}$$

and casting them into Eq. (10), which may be written in the operator form

$$J = 1 - \lambda \mathcal{L}^{-1} J \cdot g, \quad \lambda = 1, \quad \mathcal{L}^{-1} = \int_0^t (\cdot)\, \mathrm{d}s. \tag{40}$$

Equating the terms with the same power of λ yields

$$\begin{aligned} J^{(0)} &= 1, \\ J^{(h+1)} &= -\mathcal{L}^{-1} \sum_{k=0}^{h} J^{(k)} g^{(h-k)}, \quad h = 0, 1, \ldots, m-1. \end{aligned} \tag{41}$$

The sequence (41) gives the unknown terms of the decomposition (38) for J, without performing the quadrature requested by Eq. (37). Then all calculations indicated in the preceding section for derivation of the actual expressions for the probability density and the entropy function can be easily recovered.

In order to test this method, the transient behaviour of the following mechanical system with linear elastic terms, nonlinear random damping terms and two degrees of freedom is considered. The equations of motion are

$$\begin{aligned} \ddot{y} + y + r(\omega, t)\dot{z}^2 &= 0, \\ \ddot{z} + z + r(\omega, t)(\dot{y}^3 + \dot{z}^2) &= 0, \end{aligned} \tag{42}$$

with random initial conditions $y_0(\omega)$, $\dot{y}_0(\omega)$, $z_0(\omega)$ and $\dot{z}_0(\omega)$; $r(\omega, t)$ is a stochastic process coefficient given by

$$r(\omega, t) = A(\omega) \sin(\nu t - \theta(\omega)), \tag{43}$$

with A and θ independent random variables whose probability densities $P_A(A)$, $P_\theta(\theta)$ are constant with time. Introducing the augmented variable $\mathbf{x} = \{x_1 = y, x_2 = z, x_3 = \dot{y}, x_4 = \dot{z}\}^T \in \mathbf{R}^4$, the above system can be written in the form

$$\begin{aligned} \dot{x}_1 &= f_1 = x_3, \\ \dot{x}_2 &= f_2 = x_4, \\ \dot{x}_3 &= f_3 = -x_1 - A(\omega)\sin(\nu t - \theta(\omega))x_4^2, \\ \dot{x}_4 &= f_4 = -x_2 - A(\omega)\sin(\nu t - \theta(\omega))(x_3^3 + x_4^2), \end{aligned} \tag{44}$$

with initial conditions $\mathbf{x}_0 = \{x_{0,i}(\omega)\}^T$, $i = 1, \ldots, 4$ with probability density $P_0(\mathbf{x}_0)$. The approximated solution of the system (44) is obtained by decomposing the state vector $\mathbf{x}$ and the nonlinear functions f_i in the form (31 and 32), and by solving the sequence of Eqs. (33). To the fifth term of the decomposition, the resulting approximation for $\mathbf{x}(\omega, t)$ is

$$\mathbf{x}(\omega, t) \simeq \mathbf{x}_0 + \sum_{h=1}^{4} \mathbf{x}^{(h)}(\omega, t), \tag{45}$$

with

$$
\begin{aligned}
x_1^{(1)} &= tx_{0,3}, \\
x_1^{(2)} &= -t^2x_{0,1}/2 - Ax_{0,4}^2(\mathcal{L}^{-1})^2\varphi, \\
x_1^{(3)} &= -t^3x_{0,3}/6 + 2Ax_{0,2}x_{0,4}(\mathcal{L}^{-1})^2(t\varphi) + 2A^2\gamma_1x_{0,4}(\mathcal{L}^{-1})^2(\varphi\mathcal{L}^{-1}\varphi), \\
x_1^{(4)} &= t^4x_{0,1}/24 + Ax_{0,4}^2(\mathcal{L}^{-1})^4\varphi - \gamma_2(\mathcal{L}^{-1})^2(t^2\varphi) \\
&\quad - 2A^2\{\gamma_1x_{0,2}(\mathcal{L}^{-1})^2(t\varphi\mathcal{L}^{-1}\varphi) + \gamma_3x_{0,4}(\mathcal{L}^{-1})^2[\varphi\mathcal{L}^{-1}(t\varphi)]\} \\
&\quad - A^3\{\gamma_1^2(\mathcal{L}^{-1})^2[\varphi(\mathcal{L}^{-1}\varphi)^2] + 2\gamma_4x_{0,4}(\mathcal{L}^{-1})^2[\varphi\mathcal{L}^{-1}(\varphi\mathcal{L}^{-1}\varphi)]\},
\end{aligned} \tag{46a}
$$

$$
\begin{aligned}
x_2^{(1)} &= tx_{0,4}, \\
x_2^{(2)} &= -t^2x_{0,2}/2 - A\gamma_1(\mathcal{L}^{-1})^2\varphi, \\
x_2^{(3)} &= -t^3x_{0,4}/6 + A\gamma_3(\mathcal{L}^{-1})^2(t\varphi) + A^2\gamma_4(\mathcal{L}^{-1})^2(\varphi\mathcal{L}^{-1}\varphi), \\
x_2^{(4)} &= t^4x_{0,2}/24 + A[\gamma_1(\mathcal{L}^{-1})^4\varphi - \gamma_5(\mathcal{L}^{-1})^2(t^2\varphi)] \\
&\quad - 2A^2\{\gamma_6(\mathcal{L}^{-1})^2(t\varphi\mathcal{L}^{-1}\varphi) + \gamma_7(\mathcal{L}^{-1})^2[\varphi\mathcal{L}^{-1}(t\varphi)]\} \\
&\quad - A^3\{\gamma_8(\mathcal{L}^{-1})^2[\varphi\mathcal{L}^{-1}(\varphi\mathcal{L}^{-1}\varphi)] + \gamma_9(\mathcal{L}^{-1})^2[\varphi(\mathcal{L}^{-1}\varphi)^2]\},
\end{aligned} \tag{46b}
$$

$$x_3^{(h)} = \mathrm{d}x_1^{(h+1)}/\mathrm{d}t, \tag{46c}$$

$$x_4^{(h)} = \mathrm{d}x_2^{(h+1)}/\mathrm{d}t, \quad h = 1, \ldots, 4, \tag{46d}$$

where $(\mathcal{L}^{-1})^2(\cdot) = \mathcal{L}^{-1}\mathcal{L}^{-1}(\cdot)$ are easily calculated by double time-quadratures; $\varphi = \varphi(\omega, t) = \sin(\nu t - \theta(\omega))$, and $\gamma_1, \ldots, \gamma_9$ are the following functions of the initial conditions:

$$
\begin{aligned}
\gamma_1(\mathbf{x}_0) &= x_{0,3}^3 + x_{0,4}^2, \\
\gamma_2(\mathbf{x}_0) &= x_{0,2}^2 - x_{0,4}^2, \\
\gamma_3(\mathbf{x}_0) &= 3x_{0,3}x_{0,1} + 2x_{0,2}x_{0,4}, \\
\gamma_4(\mathbf{x}_0) &= x_{0,4}(3x_{0,3}^2x_{0,4} + 2\gamma_1), \\
\gamma_5(\mathbf{x}_0) &= 3x_{0,1}^2x_{0,3} - 3x_{0,3}^3/2 + \gamma_2, \\
\gamma_6(\mathbf{x}_0) &= 3x_{0,1}x_{0,3}x_{0,4}^2 + \gamma_1x_{0,2}, \\
\gamma_7(\mathbf{x}_0) &= x_{0,4}(3x_{0,2}x_{0,3}^2 + \gamma_3), \\
\gamma_8(\mathbf{x}_0) &= 2x_{0,4}(3\gamma_1x_{0,3}^2 + \gamma_4), \\
\gamma_9(\mathbf{x}_0) &= 3x_{0,3}x_{0,4}^4 + \gamma_1^2.
\end{aligned} \tag{47}
$$

Further terms in the truncated solution (45) may be obtained by straightforward calculations. However, a satisfactory approximation of the exact solution is supplied by the above five-terms decomposition, as shown by Fig. 1 where the truncated solutions

$$\zeta_n = \sum_{h=0}^{n} x_2^{(h)}, \quad n = 0, 1, 2, 3, 4,$$

for $x_2 = z(t)$ are compared with the numerical solution, by assuming $A = 0.1$, $\theta = 0$, $\nu = 1$, $x_{0,1} = x_{0,2} = 0.5$ and $x_{0,3} = x_{0,4} = 1$.

The five-terms decomposition of the Jacobian J is obtained from Eqs. (38–41) in the form

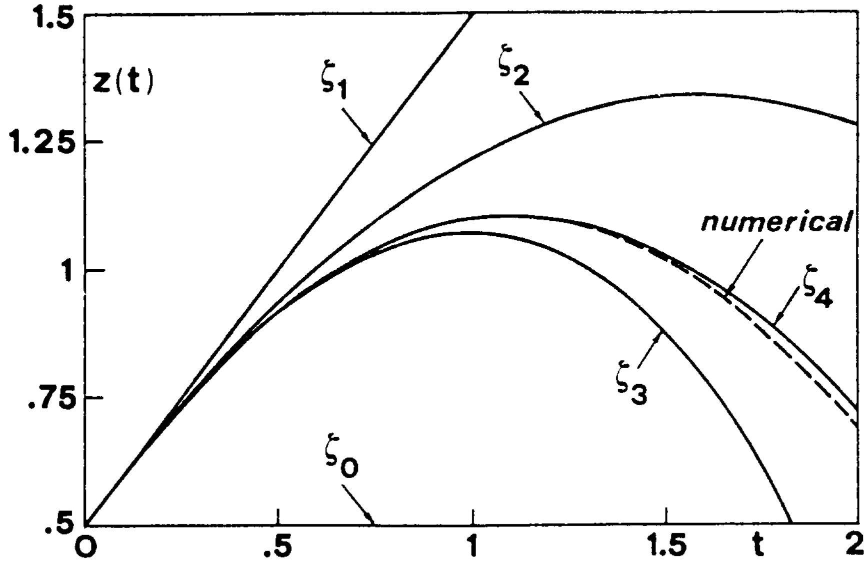

Fig. 1. Approximations for $z(t)$ and comparisons with the numerical solution: deterministic case.

$$J(t;\,\omega) \simeq \sum_{h=0}^{4} J^{(h)}(t;\,\omega), \tag{48}$$

being

$$J^{(0)} = 1;\; J^{(h+1)} = -\mathscr{L}^{-1}\sum_{k=0}^{h} J^{(k)}g^{(h-k)}, \quad h = 0,\,1,\,2,\,3, \tag{49}$$

and from Eqs. (11) and (39),

$$g^{(h-k)}(t;\,\omega) = -2A(\omega)\sin(\nu t - \theta(\omega))x_4^{(h-k)}(\omega,\,t), \tag{50}$$

where $x_4^{(h-k)}$ are given by Eq. (46). Inserting Eq. (50) into Eq. (49), one obtains

$$\begin{aligned}
J^{(1)}(t;\,\omega) &= 2Ax_{0,4}\mathscr{L}^{-1}\varphi,\\
J^{(2)}(t;\,\omega) &= -2Ax_{0,2}\mathscr{L}^{-1}(t\varphi) - 2A^2(\gamma_1 + 2x_{0,4}^2)\mathscr{L}^{-1}(\varphi\mathscr{L}^{-1}\varphi),\\
J^{(3)}(t;\,\omega) &= -Ax_{0,4}\mathscr{L}^{-1}(t^2\varphi) + 4A^2\{3x_{0,1}x_{0,3}^2\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(t\varphi)]/2\\
&\quad - x_{0,2}x_{0,4}\mathscr{L}^{-1}(t\varphi\mathscr{L}^{-1}\varphi)\} - 4A^3x_{0,4}\{\gamma_1\mathscr{L}^{-1}[\varphi(\mathscr{L}^{-1}\varphi)^2]\\
&\quad - \gamma_{10}\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(\varphi\mathscr{L}^{-1}\varphi)]\},\\
J^{(4)}(t;\,\omega) &= Ax_{0,2}\mathscr{L}^{-1}(t^3\varphi)/3 + 2A^2\{\gamma_1\mathscr{L}^{-1}[\varphi(\mathscr{L}^{-1})^3\varphi]\\
&\quad - x_{0,4}^2\mathscr{L}^{-1}(t^2\varphi\mathscr{L}^{-1}\varphi) + 2x_{0,2}^2\mathscr{L}[t\varphi\mathscr{L}^{-1}(t\varphi)]\\
&\quad - \gamma_{11}\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(t^2\varphi)]\} + 4A^3\{\gamma_{12}\mathscr{L}^{-1}\{\varphi\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(t\varphi)]\}\\
&\quad - \gamma_{13}\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(t\varphi\mathscr{L}^{-1}\varphi)] + \gamma_{14}\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}\varphi\mathscr{L}^{-1}(t\varphi)]\\
&\quad + \gamma_{15}\mathscr{L}^{-1}[t\varphi\mathscr{L}^{-1}(\varphi\mathscr{L}^{-1}\varphi)]\}\\
&\quad + 4A^4\{\gamma_{16}\mathscr{L}^{-1}\{\varphi\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}(\varphi\mathscr{L}^{-1}\varphi)]\}\\
&\quad - \gamma_{17}\mathscr{L}^{-1}\{\varphi\mathscr{L}^{-1}[\varphi(\mathscr{L}^{-1}\varphi)^2]\}\\
&\quad + \gamma_4x_{0,4}\mathscr{L}^{-1}[\varphi\mathscr{L}^{-1}\varphi\mathscr{L}^{-1}(\varphi\mathscr{L}^{-1}\varphi)]\},
\end{aligned} \tag{51}$$

with

$$
\begin{aligned}
\gamma_{10}(\mathbf{x}_0) &= x_{0,4}(3x_{0,3}^2/2 - 2x_{0,4}),\\
\gamma_{11}(\mathbf{x}_0) &= x_{0,4}^2 + \gamma_5,\\
\gamma_{12}(\mathbf{x}_0) &= 3x_{0,1}x_{0,3}^2x_{0,4} - \gamma_7,\\
\gamma_{13}(\mathbf{x}_0) &= 2x_{0,2}x_{0,4}^2 + \gamma_6,\\
\gamma_{14}(\mathbf{x}_0) &= x_{0,4}\gamma_3 + x_{0,2}\gamma_1,\\
\gamma_{15}(\mathbf{x}_0) &= x_{0,2}(\gamma_1 + 2x_{0,4}^2),\\
\gamma_{16}(\mathbf{x}_0) &= \gamma_1^2 + 2x_{0,4}\gamma_1 - \gamma_8/2 + 2\gamma_{10}x_{0,4}^2,\\
\gamma_{17}(x_0) &= 2\gamma_1 x_{0,4}^2 + \gamma_9/2.
\end{aligned}
\tag{52}
$$

The above results can now be applied to determine the probability density and the entropy function related to the solution process $\mathbf{x}(\omega, t)$. In order to develop some quantitative examples, the following two problems are considered.

Problem 1. Determine the evolution in time of the first-order probability density for the solution process $\mathbf{x} = \{y, z, \dot{y}, \dot{z}\}$ of Eq. (42), under the assumption that (i) the initial velocities $\dot{y}_0(\omega) = x_{0,3}(\omega)$ and $\dot{z}_0(\omega) = x_{0,4}(\omega)$ are independent random variables with known probability densities $P_0(x_{0,3})$, $P_0(x_{0,4})$; (ii) the initial displacements y_0, z_0 as well as the constants A, θ have deterministic values $y_0 = \bar{x}_{0,1}$, $z_0 = \bar{x}_{0,2}$ and $A = \bar{A}$, $\theta = \bar{\theta}$, respectively.

Problem 2. Determine the evolution and fluctuations of the entropy function for the system (42) under the assumption that (i) the initial conditions are deterministic: $\mathbf{x}(t = 0) = \mathbf{x}_0$; (ii) the stochastic process coefficient $r(\omega, t)$ is given by

$$
r(\omega, t) = \tfrac{1}{10}A(\omega)\sin(\nu t - \theta(\omega)),
$$

with ν deterministic and $A(\omega)$, $\theta(\omega)$ independent random variables with probability densities $P_A(A)$, $P_\theta(\theta)$.

Considering Problem 1, application of Eqs. (9) and (48) leads to

$$
P(\mathbf{x}(\omega, t); t) = \delta(x_{0,1} - \bar{x}_{0,1})\delta(x_{0,2} - \bar{x}_{0,2})P_0(x_{0,3})P_0(x_{0,4}) \sum_{h=0}^{4} J^{(h)}(t; \omega), \tag{53}
$$

where $\mathbf{x}(\omega, t) = \mathbf{x}(\bar{x}_{0,1}, \bar{x}_{0,2}, x_{0,3}(\omega), x_{0,4}(\omega), \bar{A}, \bar{\theta}; t)$ is given by the solution (45)–(47) and the terms $J^{(h)}$ are given by Eqs. (51 and 52) by setting $x_{0,1} = \bar{x}_{0,1}$, $x_{0,2} = \bar{x}_{0,2}$, $x_{0,3} = x_{0,3}(\omega)$, $x_{0,4} = x_{0,4}(\omega)$, $A = \bar{A}$ and $\theta = \bar{\theta}$. Figure 2 shows the evolution of $P(\mathbf{x}; t)$ as deduced by the above equations. The values of $P(\mathbf{x}; t)$, which are plotted versus the quantities $X = (x_1^2 + x_3^2)^{1/2}$ and $Y = (x_2^2 + x_4^2)^{1/2}$, have been calculated by assuming that the random initial velocities are beta-distributed with mean 1 and variance $\sigma^2 = 1/28$.

With regard to the Problem 2, the mean entropy function $\langle S(t)\rangle$ can be calculated, recalling Eq. (16), by

$$
\langle S(t)\rangle = -\int_{D_A}\int_{D_\theta} J(t; \mathbf{x}_0, A, \theta)\ln J(t; \mathbf{x}_0, A, \theta)P_A(A)P_\theta(\theta)\, dA\, d\theta,
$$

whereas its variance $V(t)$ is deduced from Eq. (18):

$$
V(t) = \int_{D_A}\int_{D_\theta} J^2(t; \mathbf{x}_0, A, \theta)\ln^2 J(t; \mathbf{x}_0, A, \theta)P_A(A)P_\theta(\theta)\, dA\, d\theta - \langle S(t)\rangle^2.
$$

If the values of J supplied by Eqs. (48) and (51) are inserted in the above equations, the

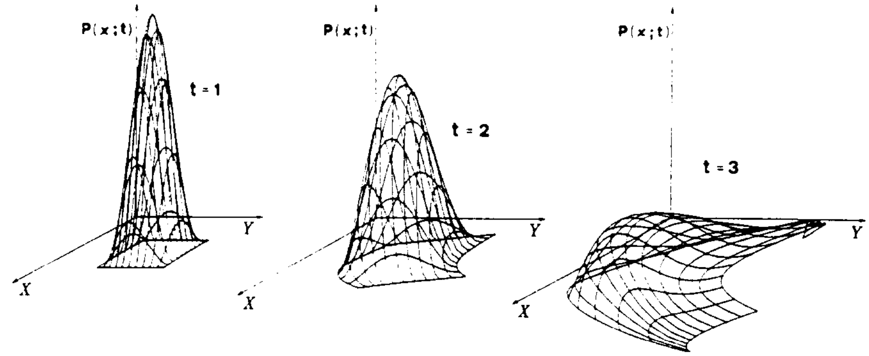

Fig. 2. Evolution of the probability density, Eq. (53), for $\bar{A} = 1$, $\bar{\theta} = 0$, $\nu = 1$ and $\bar{x}_{0,1} = \bar{x}_{0,2} = 0.5$.

approximated values of the mean and variance of the entropy function can be obtained by integration over the probability space. The resulting evolutions of $\langle S(t)\rangle$ and $V(t)$ are plotted in Fig. 3, by assuming $P_\theta(\theta) = \delta(\theta)$ and $A(\omega)$ beta-distributed with mean 1 and variance $\sigma^2 = 1/28$. By making use of the approximated solutions for $\mathbf{x}(\omega, t)$ and J, further numerical calculations may be developed without any conceptual difficulty, in order to obtain, on the grounds of the analysis made in Sec. 3, other significant statistical results concerning the considered problems as, for instance, the mean and variance of the probability density which are supplied by Eqs. (12) and (13), or the covariance between the entropy function and its time derivative, which may be recovered by application of Eq. (23).

5. DISCUSSION

This paper deals with an analysis of a large class of stochastic systems defined by nonlinear ordinary differential equations with random initial conditions and parameters. The detailed description of the considered class of systems is contained in the second section.

The paper can be split into two parts. The first part, namely Sec. 3, proposes, under suitable regularity and continuity conditions, the derivation of the evolution equation for the probability density and the entropy functions. In the same section the evolution equation for

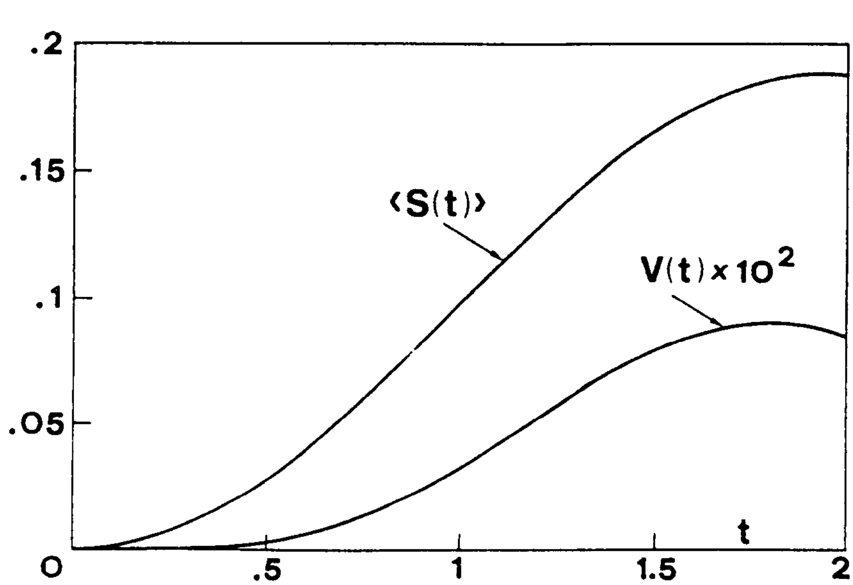

Fig. 3. Mean value and variance of the entropy function for the system (42), with $\nu = 1$, $x_{0,1} = x_{0,2} = 0.5$, $x_{0,3} = 1$ and $x_{0,4} = -1$.

the variance of the above functions is also derived so that their quantitative behaviour can be also defined. In fact the randomness of the parameters generates the fluctuations which are specified in the said section and whose quantitative analysis is certainly relevant in mathematical physics. In other words, both the probability density and the entropy function can supply a useful description of the full probabilistic state of the system. On the other hand their fluctuations can supply a description of the gap between the real behaviour of the system and its mean probabilistic state.

The second part of the paper deals with the actual mathematical methods to obtain quantitative results. In particular the Adomian decomposition method has been stressed to obtain approximate analytical solutions of the evolution equations derived in Sec. 3. The method has shown to be extremely flexible as well as accurate as clearly indicated in Fig. 1, which shows, in a deterministic calculation, how the analytical solution gets close to the numerical one with a very limited number of terms in the decomposition even for a sufficiently large time scale. The method has then shown to be very efficient also for deterministic system analysis even if the relevant advantages are evident for the stochastic equations and in particular, as developed in this paper, for the calculation of the full probability density.

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